Professor Shawn Shadden
MEC180
Mark Ansell

Assignment 5

I. Introduction

The goal of this assignment is to measure the blood flow velocity field (**u**) and use that to compute the pressure field. The problem begins by defining the Poisson's equation. This analysis is relevant to medical imaging of a human heart to aid in diagnosis of cardiac disease. Note that the math for all of the below calculations is attached.

First the Navier-Stokes equation is introduced. This governs the motion of an incompressible, Newtonian fluid.

$$(1) \quad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla P + \upsilon \nabla^2 \boldsymbol{u}$$

$$(2) \quad \nabla \cdot \boldsymbol{u} = 0$$

Equation (1) includes fluid velocity (**u**), fluid pressure (P), and kinematic viscosity (υ).

Defining b as $\upsilon \nabla^2 \boldsymbol{u} - \frac{\delta u}{\delta t} - \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and taking the divergence of equation (1) yields:

$$(3) \ \nabla \cdot (\nabla \mathrm{P} - \mathbf{b}) = 0$$

Applying Galerkin's method and integrating by parts yields:

$$(4) \quad \int_\Omega \nabla w \cdot \nabla P \, \mathrm{d}\Omega = \int_\Omega \nabla w \cdot \mathbf{b} \, \mathrm{d}\Omega + \int_\Gamma w(\nabla P - \mathbf{b}) \cdot \mathbf{n} \, \mathrm{d}\Gamma.$$

The right-hand side goes to zero due to Equation (1). The following equation is left:

$$(5) \quad \int_\Omega \nabla w \cdot \nabla P \, \mathrm{d}\Omega = \int_\Omega \nabla w \cdot \mathbf{b} \, \mathrm{d}\Omega$$

Note that Equation (5) is the weak form of a singular Poisson setup due to the lack of Dirichlet boundary conditions. To solve this, we will specify a Dirichlet boundary condition at an arbitrary boundary node.

Another analysis method involves recognizing that $\nabla \cdot \frac{\partial u}{\partial t} = 0 \ and \ \nabla \cdot \upsilon \nabla^2 \boldsymbol{u} = 0$ from Equation (2) and the commutativity of the derivative operators. The standard Galerkin procedure then yields:

$$(6) \quad \int_\Omega \nabla w \cdot \nabla P \, \mathrm{d}\Omega = \int_\Omega w \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \, \mathrm{d}\Omega + \int_\Gamma w \nabla P \cdot \mathbf{n} \, \mathrm{d}\Gamma$$

II. Method

1. Loop through velocity files
2. Define function spaces Q, V, and T for scalars, vectors, and tensors, respectively.

3. Project old velocity field to the new mesh. Need to allow extrapolation since the domain is changing.
4. Load new velocity field.
5. Project gradient of velocity to CG1 tensor space so that div(grad_velocity) can be used to approximate Laplacian of velocity.
6. Approximate time derivative of velocity using backward difference.
7. Define test, trial, and solution functions.
8. Define zero Dirichlet BC at rightmost node in mesh.
9. Define b vector.
10. Define a and L and solve.
11. Shift pressure such that all values are >= 0. This is done because PPE only gives relative pressures.

III. Results

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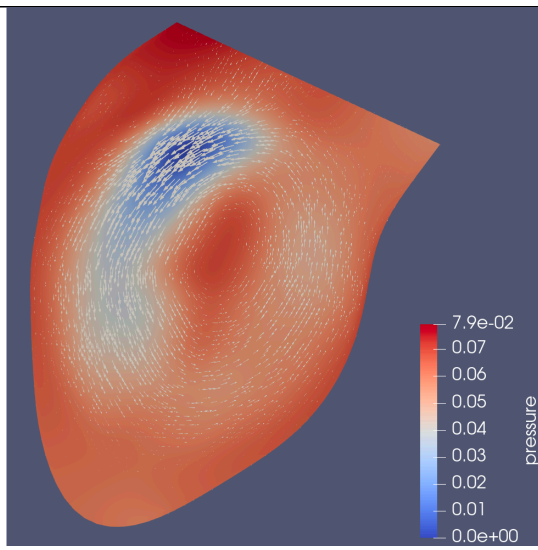

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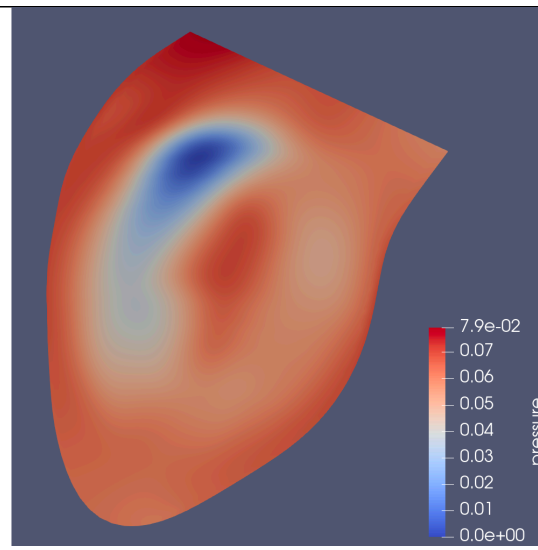

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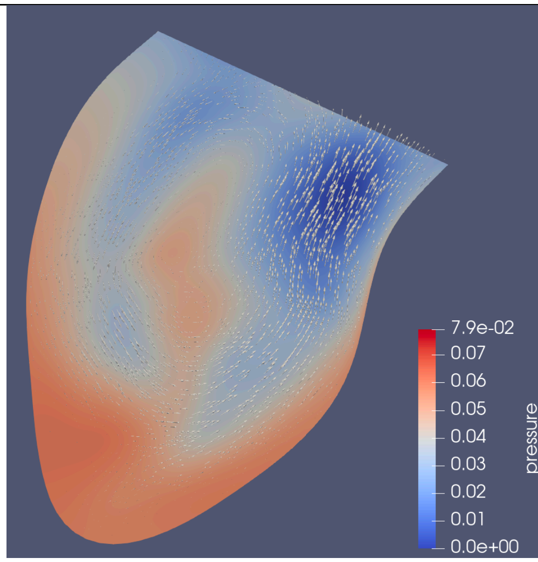
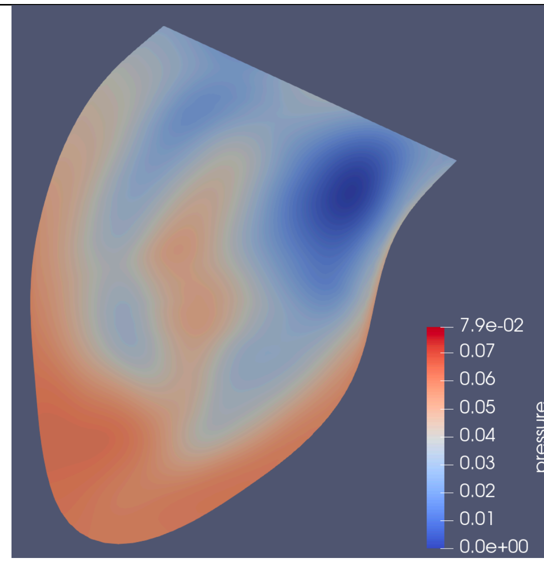
<tr><td>16</td><td>
</td><td>
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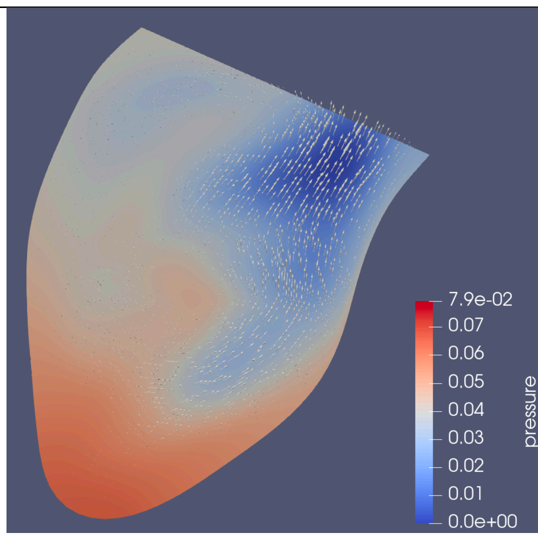
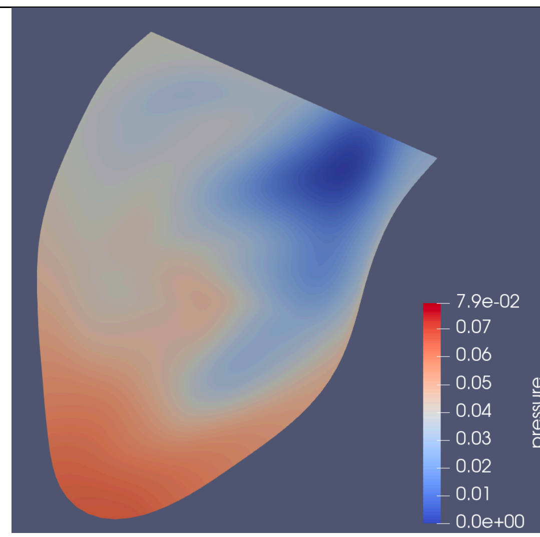
<tr><td>26</td><td>
</td><td>
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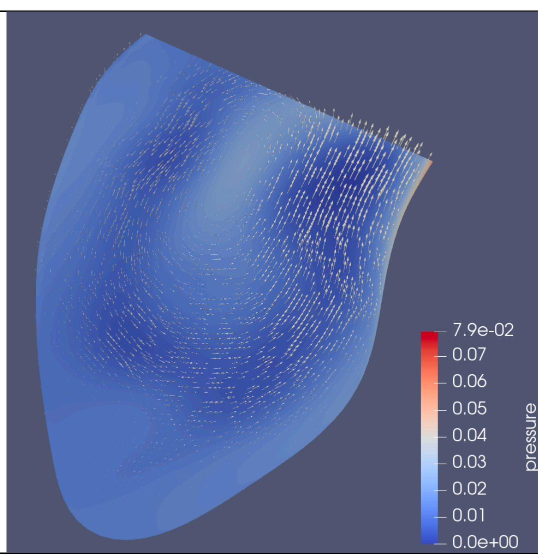
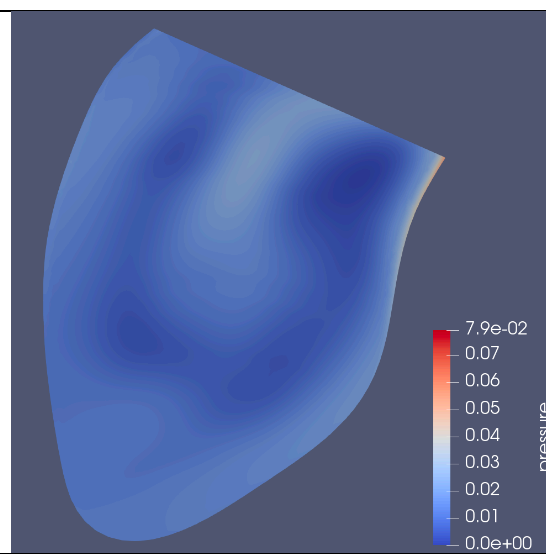
<tr><td>36</td><td>
</td><td>
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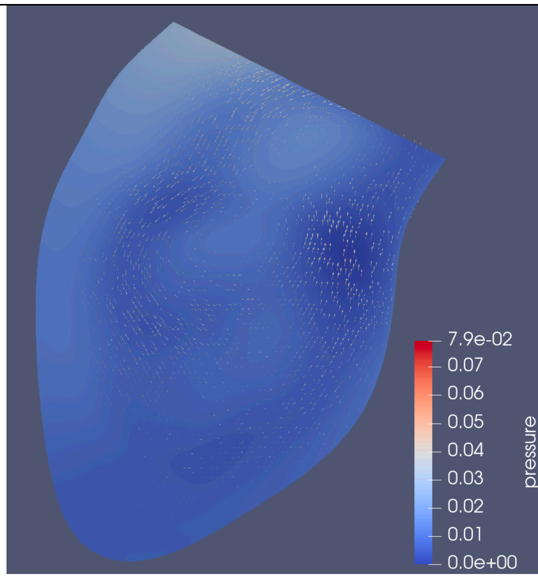
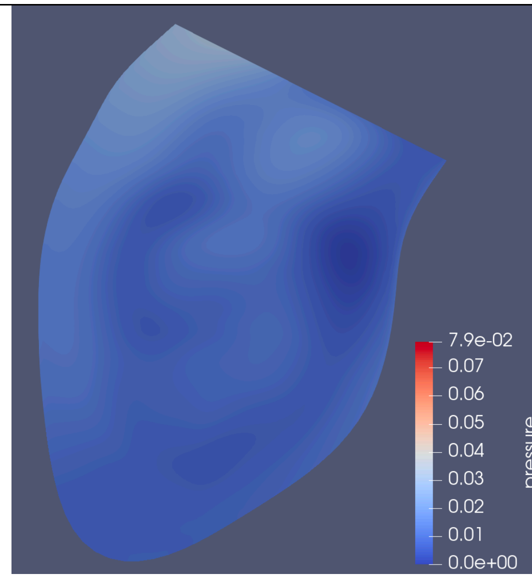
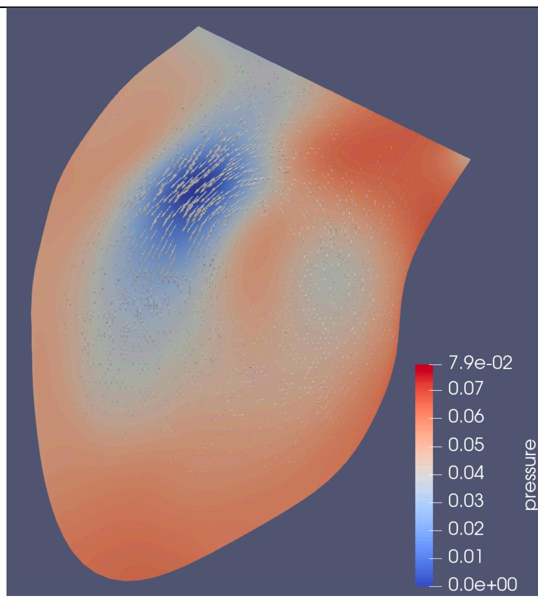
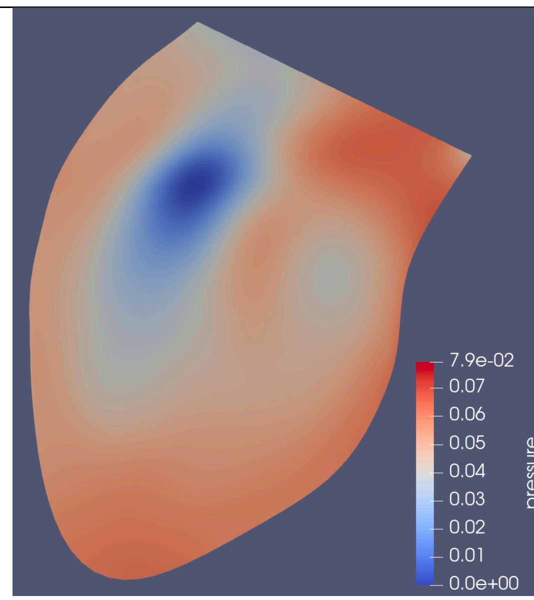
46		
56		

IV. Discussion

The video shows that velocity and pressure vary over time and space. The heart also becomes smaller before growing in size. Additionally, the viewer can notice that whenever pressure is high, velocity is low. This relationship is explained by Bernoulli’s equation, which indicates that velocity and pressure are inversely proportional.

The sequence of images also resembles blood pumping through the heart, with the heart undergoing a cyclical motion of pumping blood in and out. When pressure is high, this can be seen as resembling the heart bringing in blood. When the pressure falls lower, this can be seen as resembling the heart pumping blood out. The cyclicality is shown by the heart returning to the color red at the end of the analysis.

MEC180 Assignment 5

Navier-Stokes equation, which governs the motion of an incompressible, Newtonian fluid

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla P + \nu\nabla^2\vec{u} \quad (*)$$

$$\nabla\cdot\vec{u} = 0$$

→ u is fluid velocity
→ P is fluid pressure
→ ν is kinematic viscosity

$$b = \nu\nabla^2\vec{u} - \frac{\partial\vec{u}}{\partial t} - \vec{u}\cdot\nabla\vec{u}$$

$$\nabla\cdot(\nabla P - \vec{b})$$

Approach 1

Galerkin's Method:

(1) multiply by weighting function, w

$$\int_\Omega w\,\nabla\cdot(\nabla P - \vec{b})\,d\Omega = 0$$

$$\int_\Gamma w\cdot(\nabla P - \vec{b})\cdot n\,d\Gamma - \int \nabla w\cdot(\nabla p - \vec{b})\,d\Omega = 0$$

$$\boxed{\int_\Omega \nabla w\cdot\nabla P\,d\Omega = \int_\Omega \nabla w\cdot\vec{b}\,d\Omega + \int_\Gamma w(\nabla P - \vec{b})\cdot n\,d\Gamma} \quad (**)$$

(1) (2) (3)

(3) goes to zero due to (*)

The common finite element "do nothing" BC (zero Neumann) actually universally satisfies the normal component of the governing equation at the boundary.

(**) becomes $\int_\Omega \nabla w\cdot\nabla P\,d\Omega = \int_\Omega \nabla w\cdot\vec{b}\,d\Omega$

↳ this is the weak form of the consistent pressure Poisson equation

other approach:

Recognize that $\nabla\cdot\frac{\partial u}{\partial t} = 0$ and $\nabla\cdot\nu\nabla^2\vec{u} = 0$ and the commutativity of the derivative operators.

$$\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla P + \nu\nabla^2\vec{u}$$

$$\nabla\cdot\vec{u} = 0$$

$$\nabla\cdot\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla P + \nu\nabla^2\vec{u}\right)$$

$$\nabla\cdot\frac{\partial\vec{u}}{\partial t}\,(\to 0) + \nabla\cdot(\vec{u}\cdot\nabla\vec{u}) = -\nabla\cdot\nabla P + \nabla\cdot\nu\nabla^2\vec{u}\,(\to 0)$$

Galerkin:

$$\int_\Omega w\,\nabla\cdot(\vec{u}\cdot\nabla\vec{u})\,d\Omega = -\int_\Omega w(\nabla\cdot\nabla P)\,d\Omega$$

$$\rightarrow \int_\Omega w(\nabla\cdot\nabla P)\,d\Omega = -\int_\Omega \nabla w\cdot\nabla P\,d\Omega + \int_\Gamma w\nabla\vec{P}\cdot\hat{n}\,d\Gamma$$

$$\int_\Omega w\,\nabla\cdot(\vec{u}\cdot\nabla\vec{u})\,d\Omega = \int_\Omega \nabla w\cdot\nabla P\,d\Omega - \int_\Gamma w\nabla\vec{P}\cdot\hat{n}\,d\Gamma$$

$$\boxed{\int_\Omega \nabla w\cdot\nabla P\,d\Omega = \int_\Omega w\,\nabla\cdot(\vec{u}\cdot\nabla\vec{u})\,d\Omega + \int_\Gamma w\nabla P\cdot\hat{n}\,d\Gamma}$$